对数压力坐标系中的公式推导

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由垂直坐标转换的一般关系，得

$$\begin{cases} \dfrac{d\overrightarrow{V_h}}{dt} = -\alpha\nabla_\zeta p - \nabla_\zeta\Phi - f\vec{k}\times\overrightarrow{V_h} & (1) \\ \alpha\dfrac{\partial p}{\partial\zeta} + \dfrac{\partial\Phi}{\partial\zeta} = 0 & (2) \\ \dfrac{d}{dt}\left(ln\dfrac{\partial p}{\partial\zeta}\right) + \nabla_\zeta\cdot\overrightarrow{V_h} + \dfrac{\partial\dot{\zeta}}{\partial\zeta} = 0 & (3) \\ c_p\dfrac{dT}{dt} - \alpha\dfrac{dp}{dt} = \dot{Q} & (4) \end{cases}$$

在对数压力坐标系中，定义

$$z^* \equiv -Hln\left(\frac{p}{p_s}\right) \tag{5}$$

$$H \equiv \frac{RT_s}{g} \tag{6}$$

令 $\zeta = z^*$，则对于(1)，有

$$\frac{d\overrightarrow{V_h}}{dt} = -\alpha\nabla_{z^*}p - \nabla_{z^*}\Phi - f\vec{k}\times\overrightarrow{V_h} \tag{7}$$

由(5)，得

$$p = exp\left(lnp_s - \frac{z^*}{H}\right) \tag{8}$$

因此，p 是 z^* 的函数，故有 $\nabla_{z^*}p = 0$，则对于水平运动方程，有

$$\frac{d\overrightarrow{V_h}}{dt} = -\nabla_\zeta\Phi - f\vec{k}\times\overrightarrow{V_h} \tag{9}$$

$\zeta = z^*$ 对于(2)，有

$$\alpha\frac{\partial p}{\partial z^*} + \frac{\partial\Phi}{\partial z^*} = 0 \tag{10}$$

由(8)，得

$$\frac{\partial p}{\partial z^*} = -\frac{p}{H} \tag{11}$$

又由理想气体的状态方程 $p = \rho RT$，得

$$\frac{\partial p}{\partial z^*} = -\frac{\rho RT}{H} \tag{12}$$

将上式代入(10)，得到对数压力坐标系下的静力方程

$$\frac{\partial \Phi}{\partial z^*} = \frac{RT}{H} \tag{13}$$

$\zeta = z^*$ 对于(3)，有

$$\frac{d}{dt}\left(ln\frac{\partial p}{\partial z^*}\right) + \nabla_{z^*} \cdot \overrightarrow{V_h} + \frac{\partial \dot{z^*}}{\partial z^*} = 0 \tag{14}$$

由(11)，得

$$\frac{d}{dt}\left(ln\frac{\partial p}{\partial z^*}\right) = \frac{d}{dt}ln\left(-\frac{p}{H}\right) = \frac{d}{dt}\left(lnp\right) = \frac{1}{p}\frac{dp}{dt} \tag{15}$$

令

$$\rho_0\left(z^*\right) \equiv \rho_s exp\left(-\frac{z^*}{H}\right) \tag{16}$$

则

$$\frac{\partial \rho}{\partial z^*} = -\frac{\rho_0}{H} \tag{17}$$

故

$$\frac{w^*}{\rho_0}\frac{\partial \rho_0}{\partial z^*} = -\frac{w^*}{H} \tag{18}$$

又

$$w^* = \frac{dz^*}{dt} = \frac{d}{dt}\left(-Hln\frac{p}{p_s}\right) = -H\frac{d}{dt}\left(lnp - lnp_s\right) = -\frac{H}{p}\frac{dp}{dt} \tag{19}$$

所以

$$\frac{w^*}{\rho_0}\frac{\partial \rho_0}{\partial z^*} = \frac{1}{p}\frac{dp}{dt} = \frac{d}{dt}\left(ln\frac{\partial p}{\partial z^*}\right) \tag{20}$$

又由于

$$\nabla_{z^*} \cdot \overrightarrow{V_h} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{21}$$

$$\frac{\partial \dot{z^*}}{\partial z^*} = \frac{\partial w^*}{\partial z^*} \tag{22}$$

故(14)变为

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{\rho_0}\frac{\partial\left(\rho_0 w^*\right)}{\partial z^*}=0 \tag{23}$$

由(4)，有

$$c_p\left(\frac{\partial}{\partial t}+\overrightarrow{V_h}\cdot\nabla\right)T+c_p w\frac{\partial T}{\partial z}-\alpha\frac{dp}{dt}=\dot{Q} \tag{24}$$

两边同乘以 $\frac{R}{Hc_p}$，并注意到 $w=\frac{dz}{dt}=\frac{dz}{dp}\frac{dp}{dz}\frac{dz}{dt}=-\frac{1}{\rho g}\frac{dp}{dt}$，得

$$\left(\frac{\partial}{\partial t}+\overrightarrow{V_h}\cdot\nabla\right)\frac{RT}{H}-\frac{R}{\rho gH}\frac{dp}{dt}\frac{\partial T}{\partial z}-\frac{R}{H\rho c_p}\frac{dp}{dt}=\frac{R}{Hc_p}\dot{Q} \tag{25}$$

令 $\kappa=\frac{R}{c_p}$，并由(13)，上式可变为

$$\left(\frac{\partial}{\partial t}+\overrightarrow{V_h}\cdot\nabla\right)\frac{\Phi}{z^*}-\frac{1}{\rho}\frac{dp}{dt}\left(\frac{R}{gH}\frac{\partial T}{\partial z}+\frac{R}{Hc_p}\right)=\frac{\kappa}{H}\dot{Q} \tag{26}$$

又由(6)，(19)和理想气体状态方程，

$$-\frac{1}{\rho}\frac{dp}{dt}=-\frac{RT}{p}\frac{T_s}{T_s}\frac{g}{g}\frac{dp}{dt}=-\frac{HgT}{pT_s}\frac{dp}{dt}=w^*\frac{gT}{T_s} \tag{27}$$

则(26)变为

$$\left(\frac{\partial}{\partial t}+\overrightarrow{V_h}\cdot\nabla\right)\frac{\Phi}{z^*}+w^*\frac{gT}{T_s}\left(\frac{R}{gH}\frac{\partial T}{\partial z}+\frac{R}{Hc_p}\right)=\frac{\kappa}{H}\dot{Q} \tag{28}$$

由(6)，上式中

$$\frac{gT}{T_s}\left(\frac{R}{gH}\frac{\partial T}{\partial z}+\frac{R}{Hc_p}\right)=\frac{gT}{T_s}\left(\frac{1}{T_s}\frac{\partial T}{\partial z}+\frac{g}{T_sc_p}\right)=\frac{gT}{T_s^2}\frac{\partial T}{\partial z}+\frac{T}{T_s^2}\frac{g^2}{c_p} \tag{29}$$

由于

$$\frac{\partial T}{\partial z}=\frac{\partial T}{\partial z^*}\frac{\partial z^*}{\partial z}=\frac{\partial T}{\partial z^*}\frac{\partial z^*}{\partial p}\frac{\partial p}{\partial z}=\frac{\partial T}{\partial z^*}\frac{H\rho g}{p}=\frac{\partial T}{\partial z^*}\frac{Hg}{RT}=\frac{T_s}{T}\frac{\partial T}{\partial z^*} \tag{30}$$

故

$$\begin{aligned}\frac{gT}{T_s}\left(\frac{R}{gH}\frac{\partial T}{\partial z}+\frac{R}{Hc_p}\right)&=\frac{g}{T_s}\frac{\partial T}{\partial z^*}+\frac{T}{T_s^2}\frac{g^2}{c_p}\\&=\frac{g}{T_s}\left(\frac{\partial T}{\partial z^*}+\frac{T}{T_s}\frac{g}{c_p}\right)\\&=\frac{R}{H}\left(\frac{\partial T}{\partial z^*}+\frac{R}{H}\frac{T}{c_p}\right)\end{aligned} \tag{31}$$

由于

$$\theta = T\left(\frac{p_s}{p}\right)^{\frac{R}{c_p}} \tag{32}$$

则

$$\begin{aligned}\frac{\partial ln\theta}{\partial z^*} &= \frac{\partial}{\partial z^*}\left[lnT + \frac{R}{c_p}\left(lnp_s - lnp\right)\right] \\ &= \frac{1}{T}\frac{\partial T}{\partial z^*} - \frac{R}{c_p}\frac{\partial lnp}{\partial z^*} \\ &= \frac{1}{T}\frac{\partial T}{\partial z^*} + \frac{R}{Hc_p}\end{aligned} \tag{33}$$

因此

$$\frac{gT}{T_s}\left(\frac{R}{gH}\frac{\partial T}{\partial z} + \frac{R}{Hc_p}\right) = \frac{RT}{H}\frac{\partial ln\theta}{\partial z^*} \tag{34}$$

令静力稳定度参数

$$S = T\frac{\partial ln\theta}{\partial z^*} \tag{35}$$

以及

$$N^2 = \frac{R}{H}S \tag{36}$$

则(28)变为

$$\left(\frac{\partial}{\partial t} + \overrightarrow{V_h}\cdot\nabla\right)\frac{\Phi}{z^*} + w^* N^2 = \frac{\kappa}{H}\dot{Q} \tag{37}$$

综上，在对数压力坐标系中，基本方程组为

$$\frac{d\overrightarrow{V_h}}{dt} = -\nabla_\zeta\Phi - f\vec{k}\times\overrightarrow{V_h} \tag{38}$$

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H} \tag{39}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial\left(\rho_0 w^*\right)}{\partial z^*} = 0 \tag{40}$$

$$\left(\frac{\partial}{\partial t} + \overrightarrow{V_h}\cdot\nabla\right)\frac{\Phi}{z^*} + w^* N^2 = \frac{\kappa}{H}\dot{Q} \tag{41}$$