Equations in Spherical Cooridinates

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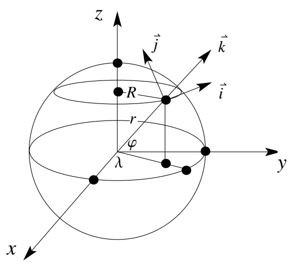
First of all, we have



$$\begin{cases} \delta x = r\cos\varphi\delta\lambda & (1) \\ \delta y = r\delta\varphi & (2) \\ \delta z = \delta r & (3) \end{cases}$$

So we get

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} \tag{4}$$

$$= \lim_{\delta x\to 0}\frac{\delta\phi}{\delta x}\vec{i} + \lim_{\delta y\to 0}\frac{\delta\phi}{\delta y}\vec{j} + \lim_{\delta z\to 0}\frac{\delta\phi}{\delta z}\vec{k}$$

$$= \lim_{\delta\lambda\to 0}\frac{\delta\phi}{r\cos\varphi\delta\lambda}\vec{i} + \lim_{\delta\varphi\to 0}\frac{\delta\phi}{r\delta\varphi}\vec{j} + \lim_{\delta r\to 0}\frac{\delta\phi}{\delta r}\vec{k}$$

$$= \frac{1}{r\cos\varphi}\frac{\partial\phi}{\partial\lambda}\vec{i} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\vec{j} + \frac{\partial\phi}{\partial r}\vec{k} \tag{5}$$

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{6}$$

$$= \left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)\cdot\left(u\vec{i} + v\vec{j} + w\vec{k}\right)$$

$$= \left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)\cdot u\vec{i}$$

$$+ \left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)\cdot v\vec{j}$$

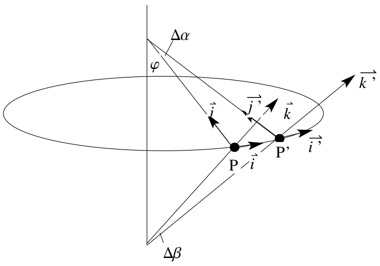
$$+ \left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)\cdot w\vec{k}$$

$$= \left(\frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda} + \vec{i}\frac{u}{r\cos\varphi}\frac{\partial\vec{i}}{\partial\lambda} + \vec{j}\frac{u}{r}\frac{\partial\vec{i}}{\partial\varphi} + \vec{k}u\frac{\partial\vec{i}}{\partial r}\right)$$

$$+ \left(\vec{i}\frac{v}{r\cos\varphi}\frac{\partial\vec{j}}{\partial\lambda} + \frac{1}{r}\frac{\partial v}{\partial\varphi} + \vec{j}\frac{v}{r}\frac{\partial\vec{j}}{\partial\varphi} + \vec{k}v\frac{\partial\vec{j}}{\partial r}\right)$$

$$+ \left(\vec{i}\frac{w}{r\cos\varphi}\frac{\partial\vec{k}}{\partial\lambda} + \vec{j}\frac{w}{r}\frac{\partial\vec{k}}{\partial\varphi} + \frac{\partial w}{\partial r} + \vec{k}w\frac{\partial\vec{k}}{\partial r}\right) \tag{7}$$

Because of

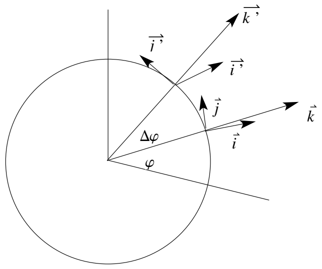


$$
\begin{aligned}
\frac{\partial \vec{i}}{\partial \lambda} &= \lim_{\Delta\lambda \to 0} \frac{\Delta \vec{i}}{\Delta \lambda} \\
&= \lim_{\Delta\lambda \to 0} \frac{\left|\Delta \vec{i}\right| \left(-\frac{\vec{R}}{R}\right)}{\Delta \lambda} \\
&= \sin\varphi \vec{j} - \cos\varphi \vec{k}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\frac{\partial \vec{j}}{\partial \lambda} &= \lim_{\Delta\lambda \to 0} \frac{\Delta \vec{j}}{\Delta \lambda} \\
&= \lim_{\Delta\lambda \to 0} \frac{\left|\Delta \vec{j}\right| \left(-\vec{i}\right)}{\Delta \lambda} \\
&= -\vec{i} \lim_{\Delta\lambda \to 0} \frac{\Delta \alpha}{\Delta \lambda} \\
&= -\vec{i} \lim_{\Delta\lambda \to 0} \frac{\widehat{PP'} / \frac{r}{\tan\varphi}}{\widehat{PP'} / \left(r\cos\varphi\right)} \\
&= -\sin\varphi \vec{i}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\frac{\partial \vec{k}}{\partial \lambda} &= \lim_{\Delta\lambda \to 0} \frac{\Delta \vec{k}}{\Delta \lambda} \\
&= \lim_{\Delta\lambda \to 0} \frac{\left|\Delta \vec{k}\right| \vec{i}}{\Delta \lambda} \\
&= \vec{i} \lim_{\Delta\lambda \to 0} \frac{\Delta \beta}{\Delta \lambda} \\
&= \vec{i} \lim_{\Delta\lambda \to 0} \frac{\widehat{PP'} / r}{\widehat{PP'} / \left(r\cos\varphi\right)} \\
&= \cos\varphi \vec{i}
\end{aligned} \tag{10}
$$

And



$$\frac{\partial \vec{i}}{\partial \varphi} = \lim_{\Delta\varphi \to 0} \frac{\Delta \vec{i}}{\Delta \varphi} = 0 \tag{11}$$

$$\begin{aligned}
\frac{\partial \vec{j}}{\partial \varphi} &= \lim_{\Delta\varphi \to 0} \frac{\Delta \vec{j}}{\Delta \varphi} \\
&= \lim_{\Delta\varphi \to 0} \frac{\left|\Delta \vec{j}\right| \left(-\vec{k}\right)}{\Delta \varphi} \\
&= -\vec{k} \lim_{\Delta\varphi \to 0} \frac{\Delta \varphi}{\Delta \varphi} \\
&= -\vec{k}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{\partial \vec{k}}{\partial \varphi} &= \lim_{\Delta\varphi \to 0} \frac{\Delta \vec{k}}{\Delta \varphi} \\
&= \lim_{\Delta\varphi \to 0} \frac{\left|\Delta \vec{k}\right| \left(\vec{j}\right)}{\Delta \varphi} \\
&= \vec{j} \lim_{\Delta\varphi \to 0} \frac{\Delta \varphi}{\Delta \varphi} \\
&= \vec{j}
\end{aligned} \tag{13}$$

And obviously

$$\frac{\partial \vec{i}}{\partial r} = \frac{\partial \vec{j}}{\partial r} = \frac{\partial \vec{k}}{\partial r} = 0 \tag{14}$$

Finally, we get

$$\nabla \cdot \overrightarrow{V} = \left(\frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda}\right) + \left(-\frac{v\sin\varphi}{r\cos\varphi} + \frac{1}{r}\frac{\partial v}{\partial\varphi}\right) + \left(\frac{w}{r} + \frac{w}{r} + \frac{\partial w}{\partial r}\right)$$
$$= \frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial\left(v\cos\varphi\right)}{\partial\varphi} + \frac{1}{r^2}\frac{\partial\left(wr^2\right)}{\partial r} \tag{15}$$

$$\nabla \times \overrightarrow{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{16}$$
$$= \frac{1}{r^2\cos\varphi}\begin{vmatrix} r\cos\varphi\vec{i} & r\vec{j} & \vec{k} \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial r} \\ r\cos\varphi u & rv & w \end{vmatrix}$$
$$= \frac{1}{r^2\cos\varphi}\left[r\cos\varphi\vec{i}\left(\frac{\partial w}{\partial\varphi} - \frac{\partial\left(vr\right)}{\partial r}\right) - r\vec{j}\left(\frac{\partial w}{\partial\lambda} - \frac{\partial\left(r\cos\varphi u\right)}{\partial r}\right) + \vec{k}\left(\frac{\partial\left(vr\right)}{\partial\lambda} - \frac{\partial\left(r\cos\varphi u\right)}{\partial\varphi}\right)\right]$$
$$= \vec{i}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi} - \frac{1}{r}\frac{\partial\left(vr\right)}{\partial r}\right) + \vec{j}\left(\frac{1}{r}\frac{\partial\left(ur\right)}{\partial r} - \frac{1}{r\cos\varphi}\frac{\partial w}{\partial\lambda}\right) + \vec{k}\left(\frac{1}{r\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{r\cos\varphi}\frac{\partial\left(u\cos\varphi\right)}{\partial\varphi}\right) \tag{17}$$

$$\nabla^2 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \tag{18}$$
$$= \left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)\cdot\left(\vec{i}\frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \vec{j}\frac{1}{r}\frac{\partial}{\partial\varphi} + \vec{k}\frac{\partial}{\partial r}\right)$$
$$= \left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{r^2\cos\varphi}\left(\vec{j}\frac{\partial\vec{i}}{\partial\varphi}\frac{\partial}{\partial\lambda} + \vec{i}\frac{\partial\vec{j}}{\partial\lambda}\frac{\partial}{\partial\varphi}\right) + \frac{1}{r\cos\varphi}\left(\vec{k}\frac{\partial\vec{i}}{\partial r}\frac{\partial}{\partial\lambda} + \vec{i}\frac{\partial\vec{k}}{\partial r}\frac{\partial}{\partial r}\right)\right]$$
$$+ \left[\frac{1}{r^2\cos\varphi}\left(\vec{j}\frac{\partial\vec{i}}{\partial\varphi}\frac{\partial}{\partial\lambda} + \vec{i}\frac{\partial\vec{j}}{\partial\lambda}\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\left(\vec{k}\frac{\partial\vec{j}}{\partial r}\frac{\partial}{\partial\varphi} + \vec{j}\frac{\partial\vec{k}}{\partial\varphi}\frac{\partial}{\partial r}\right)\right]$$
$$+ \left[\frac{1}{r\cos\varphi}\left(\vec{k}\frac{\partial\vec{i}}{\partial r}\frac{\partial}{\partial\lambda} + \vec{i}\frac{\partial\vec{k}}{\partial\lambda}\frac{\partial}{\partial r}\right) + \frac{1}{r}\left(\vec{j}\frac{\partial\vec{k}}{\partial\varphi}\frac{\partial}{\partial r} + \vec{k}\frac{\partial\vec{j}}{\partial r}\frac{\partial}{\partial\varphi}\right) + \frac{\partial^2}{\partial r^2}\right]$$
$$= \frac{1}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} - \frac{\tan\varphi}{r^2}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$$
$$= \frac{1}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{19}$$

$$\nabla^2\phi = \frac{1}{r^2\cos^2\varphi}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right)\phi + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\phi \tag{20}$$