关于 Rossby 波的相关推导

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准静力无辐散的大气方程组如下：

$$\begin{cases} \left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u-fv=-g\frac{\partial h}{\partial x} & (1)\\ \left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v+fu=-g\frac{\partial h}{\partial y} & (2)\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 & (3)\end{cases}$$

假设基本气流为均匀西风，即令

$$u=\bar{u}(y)+u' \tag{4}$$
$$v=v' \tag{5}$$
$$h=\bar{h}(y)+h' \tag{6}$$

则准静力无辐散大气方程组变为

$$\begin{cases} \left(\frac{\partial}{\partial t}+(\bar{u}+u')\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)(\bar{u}+u')-fv'=-g\frac{\partial(\bar{h}+h')}{\partial x} & (7)\\ \left(\frac{\partial}{\partial t}+(\bar{u}+u')\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)v'+f(\bar{u}+u')=-g\frac{\partial(\bar{h}+h')}{\partial y} & (8)\\ \frac{\partial(\bar{u}+u')}{\partial x}+\frac{\partial v'}{\partial y}=0 & (9)\end{cases}$$

约去二阶小量，且由于均匀西风中 $\frac{\partial \bar{u}}{\partial y}=0$，因此可以进一步化简得

$$\begin{cases} \left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'+fv'=-g\frac{\partial h'}{\partial x} & (10)\\ \left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)v'+f(\bar{u}+u')=-g\frac{\partial(\bar{h}+h')}{\partial y} & (11)\\ \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 & (12)\end{cases}$$

由于基本态中 $\bar{v}=0$，故

$$f\bar{u}=-g\frac{\partial \bar{h}}{\partial y} \tag{13}$$

则经过线性化后的大气方程组为

$$
\begin{cases}
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'-fv'=-g\frac{\partial h'}{\partial x} & (14)\\
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)v'+fu'=-g\frac{\partial h'}{\partial y} & (15)\\
\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 & (16)
\end{cases}
$$

将(14)对 y 求偏导，(15)对 x 求偏导，得

$$
\begin{cases}
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial u'}{\partial y}+\frac{\partial f}{\partial y}v'-f\frac{\partial v'}{\partial y}=-g\frac{\partial^2 h'}{\partial x\partial y} & (17)\\
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial v'}{\partial x}+\frac{\partial f}{\partial x}u'+\frac{\partial u'}{\partial x}f=-g\frac{\partial^2 h'}{\partial y\partial x} & (18)
\end{cases}
$$

由于

$$f=2\Omega sin\phi \tag{19}$$

$$\frac{\partial f}{\partial y}=\beta \tag{20}$$

$$\frac{\partial f}{\partial x}=0 \tag{21}$$

因此有

$$
\begin{cases}
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial u'}{\partial y}+\beta v'-f\frac{\partial v'}{\partial y}=-g\frac{\partial^2 h'}{\partial x\partial y} & (22)\\
\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\frac{\partial v'}{\partial x}+\frac{\partial u'}{\partial x}f=-g\frac{\partial^2 h'}{\partial y\partial x} & (23)
\end{cases}
$$

用(23)减去(22)，得

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)+\beta v'+f\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)=0 \tag{24}$$

由(16)以及 $\zeta'=\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}$，上式化为

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\zeta'+\beta v'=0 \tag{25}$$

上式是正压水平无辐散涡度方程线性化后的形式。

由于扰动速度场是水平无辐散的，故可引进扰动流函数 Ψ'，则有

$$u'=-\frac{\partial\Psi'}{\partial y} \tag{26}$$

$$v'=\frac{\partial\Psi'}{\partial x} \tag{27}$$

$$\zeta'=\nabla^2\Psi' \tag{28}$$

于是(25)化为

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\nabla^2\Psi'+\beta\frac{\partial\Psi'}{\partial x}=0 \tag{29}$$

取 β 平面近似，即令 β 为常数，并采用标准波形法，设波解为

$$\Psi'=\Psi e^{i(kx+ly-\omega t)} \tag{30}$$

则

$$\begin{cases}\dfrac{\partial\Psi'}{\partial x}=ik\Psi' & (31)\\ \dfrac{\partial\Psi'}{\partial y}=il\Psi' & (32)\\ \dfrac{\partial\Psi'}{\partial t}=-i\omega\Psi' & (33)\\ \dfrac{\partial^2\Psi'}{\partial x^2}=k^2\Psi' & (34)\\ \dfrac{\partial^2\Psi'}{\partial y^2}=l^2\Psi' & (35)\\ \nabla^2\Psi'=(k^2+l^2)\Psi' & (36)\end{cases}$$

代入(30)，得

$$(-i\omega+\bar{u}ik)(k^2+l^2)\Psi'+\beta ik\Psi'=0 \tag{37}$$

约去 $i\Psi'$，得

$$(-\omega+\bar{u}k)(k^2+l^2)+\beta k=0 \tag{38}$$

则得到频率方程

$$\omega=k\bar{u}-\frac{\beta k}{k^2+l^2} \tag{39}$$

波速

$$c_{px}=\frac{\omega}{k}=\bar{u}-\frac{\beta}{k^2+l^2} \tag{40}$$

波群速

$$c_{gx}=\frac{\partial\omega}{\partial k}=\bar{u}+\frac{\beta(k^2-l^2)}{k^2+l^2} \tag{41}$$