Dynamic Meteorology

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Q1: French scientists have developed a high-altitude balloon that remains at constant potential temperature as it circles the earth. Suppose such a balloon is in the lower equatorial stratosphere where the temperature is isothermal at $200K$. If the balloon were displaced vertically from its equilibrium level by a small distance δz it would tend to oscillate about the equilibrium level. What is the period of this oscillation?

A1: When the ballon moves a small vertical distance δz, the acceleration is

$$\frac{d^2\delta z}{dt^2} = \frac{\rho_{air} - \rho_{ballon}}{\rho_{ballon}} g \tag{1}$$

because of $p = \rho RT$, we get

$$\frac{d^2\delta z}{dt^2} = \frac{\frac{p_{air}}{RT_{air}} - \frac{p_{ballon}}{RT_{ballon}}}{\frac{p_{ballon}}{RT_{ballon}}} g = \frac{T_{ballon} - T_{air}}{T_{air}} g \tag{2}$$

where

$$\begin{cases} T_{air} = T_0 + \dfrac{\partial T}{\partial z}\delta z = T_0 - \Gamma\delta z & (3) \\ T_{ballon} = T_0 + \dfrac{dT}{dz}\delta z = T_0 - \Gamma_d\delta z & (4) \end{cases}$$

so

$$\begin{aligned} \frac{d^2\delta z}{dt^2} &= \frac{\Gamma - \Gamma_d}{T_0 - \Gamma\delta z} g\delta z \\ &= \frac{\Gamma - \Gamma_d}{T_0\left(1 - \frac{\Gamma\delta z}{T_0}\right)} g\delta z \\ &\approx \frac{g\delta z}{T_0}\left(1 + \frac{\Gamma\delta z}{T_0}\right)(\Gamma - \Gamma_d) \\ &\approx \frac{g\delta z}{T_0}(\Gamma - \Gamma_d) \end{aligned} \tag{5}$$

and we set

$$\frac{d^2\delta z}{dt^2} = -N^2\delta z \tag{6}$$

$$N^2 = g\frac{\Gamma_d - \Gamma}{T} \tag{7}$$

where Γ is actually Γ_{air} which means the adiabatic lapse rate of the environment, and $\Gamma_{air} = 0$. So,

$$N^2 = g\frac{\Gamma_d}{T} = \frac{9.8^2 \times 10^{-3}}{200}$$
$$N = 9.8\sqrt{\frac{1}{200 \times 10^3}}\ s^{-1} \tag{8}$$

Finally, we get the period of the oscillation

$$\tau = \frac{2\pi}{N} \approx 286.58\ s \tag{9}$$

Q2: Show that for an atmosphere with an adiabatic lapse rate (i.e., constant potential temperature) the geopotential height is given by

$$Z = H_\theta \left[1 - (p/p_0)^{R/c_p}\right]$$

where p_0 is the pressure at $Z = 0$ and $H_\theta \equiv c_p\theta/g_0$ is the total geopotential height of the atmosphere.

A2: As the equation of static equilibrium goes,

$$\frac{dp}{dz} = -\rho g \tag{10}$$

Because of $p = \rho RT$,

$$\frac{dp}{dz} = -\frac{p}{RT}g$$
$$\Rightarrow\ \frac{dlnp}{dz} = -\frac{g}{RT}$$
$$\Rightarrow\ dz = -\frac{RT}{g}dlnp \tag{11}$$

and because of

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \tag{12}$$

we get

$$dz = -\frac{R\theta\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}}{g}dlnp$$
$$= -\frac{R\theta p^{\frac{R}{c_p}-1}}{gp_0^{\frac{R}{c_p}}}dp \tag{13}$$

integrate the equation above

$$\int_0^Z dz = \int_{p_0}^{p} -\frac{R\theta p^{\frac{R}{c_p}-1}}{g p_0^{\frac{R}{c_p}}} dp \tag{14}$$

if we see g as a constant and name it as g_0, we get

$$\begin{aligned} Z &= -\frac{R\theta}{g_0 p_0^{\frac{R}{c_p}}} \int_{p_0}^{p} p^{\frac{R}{c_p}-1} dp \\ &= -\frac{c_p\theta}{g_0 p_0^{\frac{R}{c_p}}} \left(p^{\frac{R}{cp}} - p_0^{\frac{R}{c_p}} \right) \\ &= -\frac{c_p\theta}{g_0} \left[\left(\frac{p}{p_0} \right)^{\frac{R}{cp}} - 1 \right] \end{aligned} \tag{15}$$

if we set

$$H_\theta = c_p\theta/g_0 \tag{16}$$

we can finally get

$$Z = H_\theta \left[1 - (p/p_0)^{R/c_p} \right] \tag{17}$$